Mapping Class Group of a Torus

And a Word on its Application

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Preliminaries

Definition

Let S be an orientable surface. The **mapping class group** of S, denoted by MCG(S), is the group of homotopy classes of orientation-preserving homeomorphisms of S, i.e. MCG$(S) = \text{Homeo}^+(S)\ /\ \text{Homeo}_0(S)$.

Definition

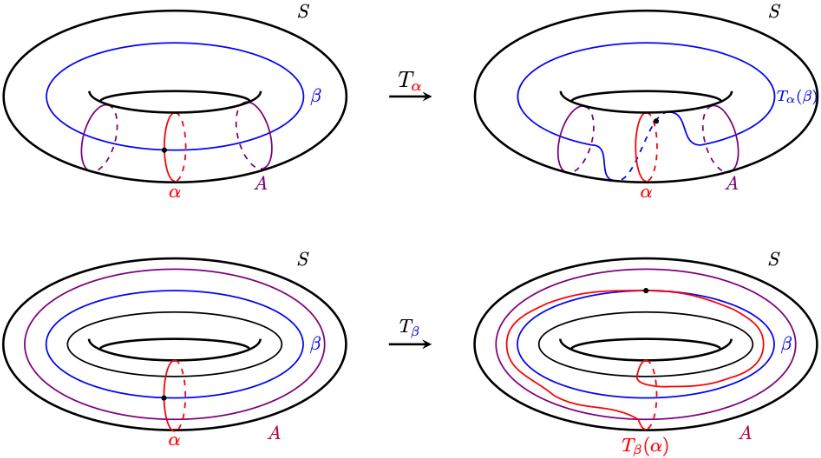
We define a **Dehn twist** on an annulus A as follows:

$$\begin{aligned} T_A : A &\longrightarrow A \\ (r, \theta) &\longmapsto (r, \theta - 2\pi r), \end{aligned}$$

where the boundary of A, denoted ∂A, is fixed pointwise.

Let S be an orientable surface with two simple closed curves α and β. Then a Dehn twist about α on S is obtained by choosing an annulus A, applying T_α and extending by the identity, i.e. fixing every point in $S \setminus A$.

Dehn Twists on a Torus



Mapping Class Group of a Torus

Theorem

$MCG(T^2) \cong SL(2, \mathbb{Z})$.

Proof

Outline:

1. Construct a map $\sigma : \mathrm{MCG}(T^2) \to \mathrm{SL}(2, \mathbb{Z})$.
2. Take an element of $\mathrm{MCG}(T^2)$. In particular, let T_α be a Dehn twist about a simple closed curve α on T^2.
3. Using the fact that $\mathbb{R}^2$ is the universal cover of T^2, choose a preferred lift $\tilde{T}_\alpha$ of the mapping class representative.
4. Every simple closed curve on a torus can be homotoped to intersect a point and lifts to a line through the origin which also passes through another integer point. In fact, the first such point is (n, m), where $\gcd(n, m) = 1$.

Mapping Class Group of a Torus

Theorem

$MCG(T^2) \cong SL(2, \mathbb{Z})$.

Proof. (cont)

6. Since there is a bijective correspondence between nontrivial homotopy classes of oriented simple closed curves on T^2 and the primitive elements of Z^2, there must exist some matrix $A \in \mathrm{SL}(2, \mathbb{Z})$ such that $A((n, m)) = (1, 0)$.
7. Notice that $\tilde{T}_\alpha(1, 0) = (1, 0)$ and $\tilde{T}_\alpha(0, 1) = (1, 1)$. Thus, $\tilde{T}_\alpha$ is a linear, orientation-preserving homeomorphism of $\mathbb{R}^2$ preserving $\mathbb{Z}^2$.
8. It follows that $\tilde{T}_\alpha$ is isomorphic to T_α, where $T_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that $T_\alpha(1, 0) = (1, 0)$ and $T_\alpha(0, 1) = (1, 1)$.
9. So we can represent a Dehn twist about α as

$$T_\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Mapping Class Group of a Torus

Theorem

$MCG(T^2) \cong SL(2, \mathbb{Z})$.

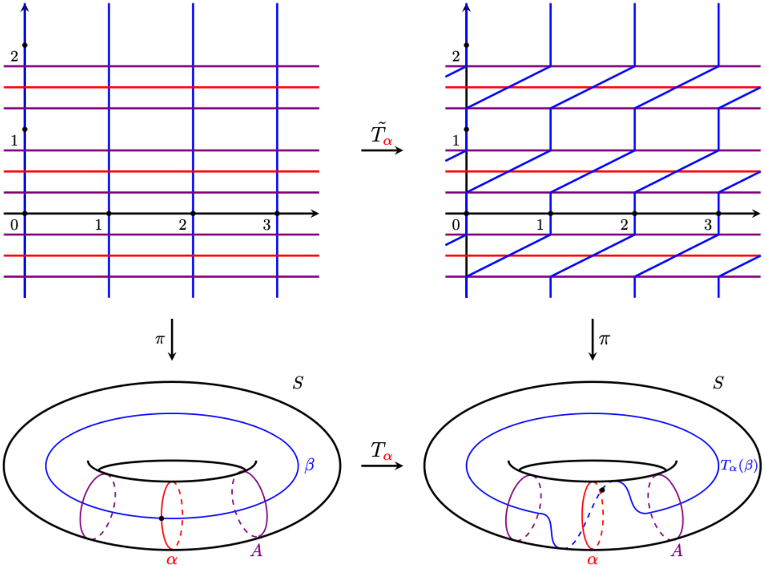
Proof. (cont)

11. Similarly, we find that we can represent another representative of a mapping class, a Dehn twist about β, as

$$T_\beta = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

12. Recall that $\mathrm{SL}(2, \mathbb{Z})$ is the set of all 2×2 matrices with integer entries and determinant 1 that is generated by the matrices T_α and T_β. It turns out that the mapping class group of the torus is generated by the same matrices. $\square$

Mapping Class Group of a Torus



A Word on its Application

Theorem (Nielsen-Thurston Classification)

Every homeomorphism between compact orientable surfaces is either periodic, reducible or psuedo-Anosov.

Remark

Mapping class groups act on Teichmüller space by isometries. So MCG(T^2) acts on Teich$(T^2) \cong \mathbb{H}^2$ by isometries. An isometry of the hyperbolic plane is a Möbius transformation γ. If we study the fixed points of γ, then we obtain the three cases of the NTC. In the Anosov case, γ has two distinct fixed points on $\mathbb{R} \cup \{\infty\}$.

Upshot

We can extend our constructions of Anosov homeomorphisms of T^2 to psuedo-Anosov homeomorphisms on S_g, where $g \geq 2$.